$$R(\mathbf{v},\mathbf{v}';\mathbf{k},\mathbf{k}')=k_D{}^2\frac{\omega+\mathbf{k}\cdot\mathbf{v}}{k'^2+k_D{}^2}\frac{(\mathbf{k}+\mathbf{k}')\cdot\mathbf{v}}{\omega+(\mathbf{k}+\mathbf{k}')\cdot\mathbf{v}}+k_D{}^2\frac{\mathbf{k}'\cdot\mathbf{v}'(\mathbf{k}+\mathbf{k}')\cdot\mathbf{v}}{k'^2(\omega+(\mathbf{k}+\mathbf{k}')\cdot\mathbf{v})}+\frac{\omega_p{}^2}{k'^2+k_D{}^2}\omega\frac{\mathbf{k}'\cdot(\mathbf{k}+\mathbf{k}')}{(\omega+(\mathbf{k}+\mathbf{k}')\cdot\mathbf{v})^2}+(\text{preceding terms with }\mathbf{k}\leftrightarrow\mathbf{k}',\ \mathbf{v}\leftrightarrow\mathbf{v}')$$

$$\Psi_1(z)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{du}{u-z}\bar{S}(u;\mathbf{k},\mathbf{k}')$$

$$\Psi_2(z)=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{du}{u-z}(k/k')\bar{S}(-(\omega+kv)/k';\mathbf{k}',\mathbf{k})$$

$$\Psi_{1,2}{}^{\pm}(u)=\lim_{\epsilon\to 0}\Psi_{1,2}(u\pm i\epsilon).$$

The "barring" operation is the multiplication by $\delta(u-\hat{k}\cdot\mathbf{v})$ and application of $\int d\mathbf{v}$. The contour C is a line from $-\infty$ to $+\infty$ an infinitesimal distance below the real axis. This notation agrees essentially with that introduced by Guernsey. The algebra used to arrive at (A1) and (A2) is tedious but straightforward.

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Interaction of Electromagnetic Waves with Quantum and Classical Plasmas*

Amiram Ron†
Plasma Physics Laboratory, Princeton University, Princeton, New Jersey

AND

Narkis Tzoar
Bell Telephone Laboratories, Whippany, New Jersey

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A systematic study of the absorption of electromagnetic waves in a quantum (classical) plasma is given, for waves whose frequencies are high compared to the collision frequency and whose wavelengths are long compared to the Bohr (Debye) radius. The treatment rests on the introduction of the temperature-dependent Green's function and Kubo's formula for the conductivity. An exact expression for the conductivity is obtained for a quantum plasma, which in its classical limit is in complete agreement with Dawson and Oberman, and with Oberman, Ron, and Dawson. The special case is treated of a quantum system of electrons in the presence of fixed ion scatterers.

I. INTRODUCTION

Recently some calculations of the absorption of electromagnetic waves in a plasma have been given. The absorption in classical plasmas has been treated with an elementary model by Dawson and Oberman[1] and by Oberman, Ron and Dawson[2] via the Liouville hierarchy. The latter work gives a complete classical derivation of the high-frequency conductivity of a plasma taking into account properly collective effects. Another approach to the *classical* problem has been given by Perel and Eliashberg[3] who begin with a quantum-mechanical diagram technique, but pass to the classical limit, before obtaining any results. Their procedure from the beginning is asymmetrical in the treatment of ions and electrons (they include the ions only in the dielectric function and neglect their *direct* contribution to the current). Their further limiting procedure in letting the ion mass become infinite is ambiguous, and it is not clear from their article to what degree of ion correlation their result is to apply. (See Discussion.) A similar approach to the same problem has been given recently by DuBois, Gilinksy, and Kivelson.[4] Their results disagree with those of references 1 and 2 and, hence, with those of the present work, and we believe because of the nonsystematic omission of a certain class of diagrams. *Note added in proof.* At the present authors' suggestion this omission has been corrected and the results incorporated in their published version [Phys. Rev. **129**, 2376 (1963)].

The purpose of the present paper is to study the absorption problem in *both classical* and *quantum* plasmas. Perel and Eliashberg[3] study the problem beginning with

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† On leave of absence from Technion; IIT, Haifa, Israel.
[1] J. Dawson and C. Oberman, Phys. Fluids **5**, 517 (1962).
[2] C. Oberman, A. Ron and J. Dawson, Phys. Fluids **5**, 1514 (1962).
[3] V. I. Perel and G. M. Eliashberg, Zh. Eksperim. i Teor. Fiz. **41**, 886 (1961) [translation: Soviet Phys.—JETP **14**, 633 (1962)]. (Hereafter referred to as PE.)
[4] D. F. Dubois, V. Gilinsky, and M. G. Kivelson, Phys. Rev. Letters **8**, 419 (1962). AEC Report, RM-3224-AEC, 1962 (unpublished). We are indebted to these authors for sending us a copy of their work before publication.

Kubo's[5] expression for the conductivity in terms of the autocorrelation function of the current operators and then use the temperature-dependent Green's function method; however, we generalized their treatment to a *full* quantum plasma (not only the semiclassical limit) of multispecie systems. We obtain the leading asymptotic contribution to the complex conductivity in a quantum (classical) plasma, when the number of particles in the Bohr (Debye) sphere is large, the frequency is higher than the collision frequency, and the wavelength of the incident field is larger than the Bohr (Debye) radius. Furthermore, this expression for the conductivity is valid for all temperatures, and is in complete agreement with the classical results of references (1) and (2).

Section II deals with the well-known relation between Kubo's formula for the conductivity and the temperature-dependent Green's function. We employ the diagram technique of Luttinger and Ward[6] in Sec. III to obtain our general result for the absorption coefficient. In Sec. IV we discuss some special cases, namely the classical multispecie system, a semiclassical hydrogenic plasma with infinitly heavy ions (taking into account their thermal equilibrium correlations), and a quantum plasma with random distributed fixed ions (impurity scatterers). We conclude our paper, Sec. V, with a brief discussion of the results.

II. GENERAL FORMULATION OF THE PROBLEM

We start from the general expression of the conductivity for a system of charged particles as given by Kubo[4]

$$\sigma_{\mu\nu}(\mathbf{k},\omega)=\frac{1}{V}\int_0^\infty d\tau\, e^{i\omega\tau}\times\int_0^\beta d\lambda\langle j_\mu(\mathbf{k},\,\tau-i\hbar\lambda)j_\nu(-\mathbf{k},\,0)\rangle, \quad (1)$$

where $\mathbf{k}$ and ω are the wave vector and the frequency of the electromagnetic wave,

$$\mathbf{j}(\mathbf{k},t)=e^{iHt/\hbar}\mathbf{j}(\mathbf{k},0)e^{-iHt/\hbar} \quad (2)$$

is the Fourier transform of the current operator in the Heisenberg representation, and the average of an operator o is given by

$$\langle o\rangle=\mathrm{Tr}\{e^{\beta(\Omega+\Sigma_s\mu_sN_s-H)}o\}. \quad (3)$$

In Eqs. (2) and (3), H represents the total Hamiltonian of the system, Ω, is defined by

$$e^{-\beta\Omega}=\mathrm{Tr}\{e^{\beta(\Sigma_s\mu_sN_s-H)}\}, \quad (4)$$

where μ_s and N_s are, respectively, the chemical potential and the number operator of the s species in the system, and β the inverse of the temperature in energy units. We use the following convention for the Fourier transforms:

$$f(\mathbf{x},t)=\frac{1}{2\pi}\int d\omega\frac{1}{V}\sum_{\mathbf{k}} e^{-i\omega t-i\mathbf{k}\cdot\mathbf{x}}f(\mathbf{k},\omega) \quad (5)$$

and

$$f(\mathbf{k},\omega)=\int dt\int d\mathbf{x}\, e^{i\omega t+i\mathbf{k}\cdot\mathbf{x}}f(\mathbf{x},t).$$

In order to render Eq. (1) in a more convenient form we integrate it by parts and obtain

$$\sigma_{\mu\nu}(\mathbf{k},\omega)=\sigma_{\mu\nu}^{(1)}(\mathbf{k},\omega)+\sigma_{\mu\nu}^{(2)}(\mathbf{k},\omega), \quad (6)$$

where

$$\sigma_{\mu\nu}^{(1)}(\mathbf{k},\omega)=-\frac{1}{i\omega}\frac{1}{V}\int_0^\beta d\lambda\langle j_\mu(\mathbf{k},\,-i\hbar\lambda)j_\nu(-\mathbf{k},\,0)\rangle =i\frac{1}{4\pi\omega}\sum_s\omega_s^2=\sigma_0(\omega). \quad (7)$$

$\omega_s^2=4\pi e_s^2n_s/m_s$ is the plasma frequency, e_s is the charge, n_s is the average particle density and m_s is the particle mass of the s species, and

$$\sigma_{\mu\nu}^{(2)}(\mathbf{k},\omega)=\frac{1}{\hbar\omega}\frac{1}{V}\int_0^\infty d\tau\, e^{i\omega\tau}\langle[j_\mu(\mathbf{k},\tau),\,j_\nu(-\mathbf{k},\,0)]\rangle. \quad (8)$$

In Eq. (8) [,] denotes the commutator.

In order to facilitate the temperature-dependent Green's function formalism[7] we define a function

$$Y_{\mu\nu}(\mathbf{k},z)=\frac{1}{\hbar}\int_{-\infty}^\infty\frac{d\omega'}{\omega'-z}(1-e^{-\hbar\omega'\beta})\Phi_{\mu\nu}(\mathbf{k},\omega'), \quad (9)$$

which is analytic in the entire z plan, except for a cut on the real axis. $\Phi_{\mu\nu}(\mathbf{k},\omega)$ is real and given by

$$\Phi_{\mu\nu}(\mathbf{k},\omega)=\frac{1}{V}\sum_{m,n}\exp[\beta(\Omega+\textstyle\sum_s\mu_sN_s^{(n)}-E_n)]\times\langle n|j_\mu(\mathbf{k},0)|m\rangle\langle m|j_\nu(-\mathbf{k},\,0)|n\rangle\times\delta\!\left(\omega-\frac{E_m-E_n}{\hbar'}\right), \quad (10)$$

where m and n represent eigenstates of the Hamiltonian and the number operator, with

$$H|n\rangle=E_n|n\rangle,\quad N_s|n\rangle=N_s^{(n)}|n\rangle, \quad (11)$$

and $N_s^{(n)}=N_s^{(m)}$ in Eq. (10) due to the fact that $\mathbf{j}$ commutes with the number operator. If we represent explicitly the average in Eq. (8) as a sum over states [see Eq. (10)] and use the fact that $N_s^{(n)}=N_s^{(m)}$ we

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[5] R. Kubo, J. Phys. Soc. Japan **12**, 570 (1957).

[6] J. M. Luttinger and J. C. Ward, Phys. Rev. **118**, 1417 (1960). (Hereafter referred to as LW.)

[7] A. A. Abrikosov, L. P. Gorkov, and I. E. Dzyaloshinsky, Zh. Eksperim. i Teor. Fiz. **36**, 900 (1959) [translation: Soviet Phys.—JETP **9**, 636 (1959)]. Many other references can be found in reference 6.

obtain

$$\sigma_{\mu\nu}(\mathbf{k},\omega)=\frac{1}{i\omega}Y_{\mu\nu}{}^{+}(\mathbf{k},\omega), \tag{12}$$

where for any function $f(z)$ in the complex z plane, we denote by

$$f^{\pm}(\omega)=\lim_{z\to\omega\pm i\epsilon} f(z) \qquad \epsilon\to+0. \tag{13}$$

In order to obtain the function $Y(\mathbf{k},z)$ of Eq. (8) we define a Green's function

$$M_{\mu\nu}(\mathbf{k},u)=\frac{1}{V}\langle T\{j_\mu(\mathbf{k},u)j_\nu(-\mathbf{k},0)\}\rangle;\ -\beta<u<\beta, \tag{14}$$

where T is the Dyson ordering operator and

$$\mathbf{j}(\mathbf{k},u)=e^{uH}\mathbf{j}(\mathbf{k},0)e^{-uH}. \tag{15}$$

By expressing our Green's function $M(\mathbf{k},u)$ in terms of the sum over states, as in Eq. (10), one easily convinces himself that $M(\mathbf{k},u)$ is a periodic function in u,

$$M_{\mu\nu}(\mathbf{k},u+\beta)=M_{\mu\nu}(\mathbf{k},u).$$

Thus, its "Fourier transform" with respect to u,

$$M_{\mu\nu}(\mathbf{k},n)=\int_0^\beta du\, e^{i2\pi nu/\beta}M_{\mu\nu}(\mathbf{k},u) \qquad n=0,\pm1,\pm2,\cdots, \tag{16}$$

enjoys the property

$$M_{\mu\nu}(\mathbf{k},n)=\frac{1}{\hbar}\int\frac{d\omega'}{\omega'-i2\pi n/\beta\hbar}(1-e^{-\beta\hbar\omega'})\Phi_{\mu\nu}(\mathbf{k},\omega'). \tag{17}$$

If we now compare Eqs. (17) and (9) we realize that $Y(\mathbf{k},z)$ is the analytical continuation of $M(\mathbf{k},n)$ from the infinite set of points $i2\pi n/\hbar\beta$ ($n>0$) on the positive imaginary axis of z to the entire upper half-plane of z. In conclusion we see that our problem reduces to the calculation of $M(\mathbf{k},n)$ which will be evaluated using the well-known Green's function diagram technique.

III. EVALUATION OF THE ABSORPTION COEFFICIENT

We turn now to the calculation of $M(\mathbf{k},n)$ using perturbation expansion, and then resumming *all* diagrams (terms) which contribute to the conductivity for quantum (classical) plasma, when the number of particles in the Bohr (Debye) sphere is large, the frequency is high compared to the collision frequency, and the wavelength of the incident field is large compared to the Bohr (Debye) radius. Thus, in resumming the diagrams (terms) of the perturbation expansion, we consider processes proportional to the number of particles, N, as finite and include them to all orders while those processes which are not proportional to N are treated as small. This point has been discussed in detail by Balescu.[8]

We write now our Green's function in the second quantization formalism in the interaction representation as

$$M_{\mu\nu}(\mathbf{k},u)=\frac{\hbar^2}{V}\sum_{s,s'}\frac{e_se_{s'}}{m_sm_{s'}}\sum_{\mathbf{p},\mathbf{p}'}p_\mu p_\nu\langle U(\beta)\rangle_0^{-1}$$
$$\times\langle T\{a_{\mathbf{p}+\mathbf{k}/2}{}^{+}(s,u)a_{\mathbf{p}-\mathbf{k}/2}(s,u)$$
$$\times a_{\mathbf{p}'-\mathbf{k}/2}{}^{+}(s',0)a_{\mathbf{p}'+\mathbf{k}/2}(s',0)U(\beta)\}\rangle_0, \tag{18}$$

where use has been made of the second quantization representation of the current operator

$$\mathbf{j}(\mathbf{k})=\hbar\sum_s\frac{e_s}{m_s}\sum_{\mathbf{p}}\mathbf{p}a_{\mathbf{p}+\mathbf{k}/2}{}^{+}(s)a_{\mathbf{p}-\mathbf{k}/2}(s). \tag{19}$$

The symbol $\langle\ \rangle_0$ corresponds to the average defined in Eq. (3) for noninteracting particles,

$$U(\beta)=\exp\left\{-\int_0^\beta du\, H_I(u)\right\}, \tag{20}$$

and

$$H_I=\frac{1}{2V}\sum_k\frac{4\pi}{k^2}\sum_{s,s'}e_se_{s'}\sum_{\mathbf{p},\mathbf{p}'}a_{\mathbf{p}+\mathbf{k}/2}{}^{+}(s)a_{\mathbf{p}'-\mathbf{k}/2}{}^{+}(s')$$
$$\times a_{\mathbf{p}'+\mathbf{k}/2}(s')a_{\mathbf{p}-\mathbf{k}/2}(s). \tag{21}$$

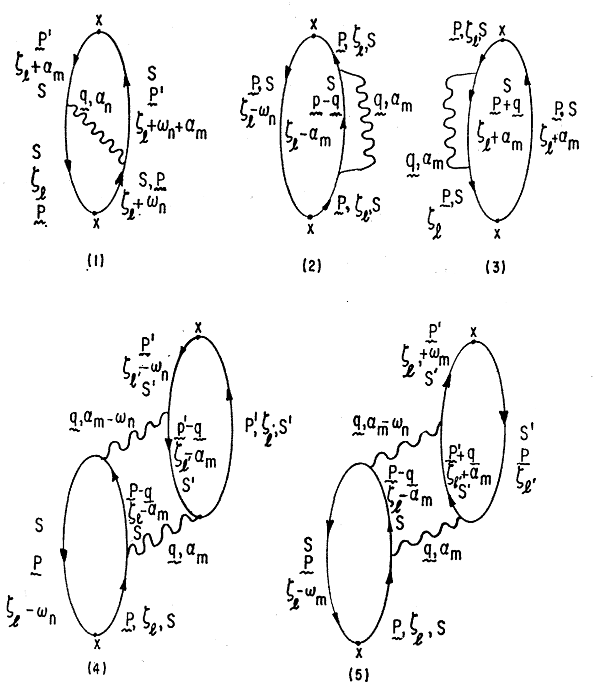

FIG. 1. The class of diagrams which contribute to the high-frequency conductivity.

[8] R. Balescu, Phys. Fluids 4, 94 (1961). Many references concerning quantum and classical plasmas can be found there.

The basic rules for the perturbation expansion of $M(\mathbf{k},n)$ are given by Luttinger and Ward.[6] We use their diagram technique and indicate by a solid line with labels s, $\mathbf{p}$, and ζ_l the s-species free-particle propagator of wave vector $\mathbf{p}$ and "energy" ζ_l (we restrict ourself to fermions only)

$$G_{\mathbf{p}}(\zeta_l,s)=[\zeta_l(s)-\epsilon_p(s)]^{-1},$$
$$\zeta_l(s)=i\pi(2l+1)\frac{1}{\beta}+\mu_s;\quad l=0,\pm1,\pm2,\cdots,\qquad(22)$$
$$\epsilon_p(s)=\frac{\hbar^2p^2}{2m_s},$$

and by dotted line the interaction $4\pi/k^2$. To each vertex we assign a charge e_s given by the s label of the particle.

In the high-frequency long-wavelength region we take into account a generalized version of the diagrams considered by Perel and Eliashberg.[3] Our generalization corresponds to considering all species in equivalent manner, see Fig. 1. The wavy line represents the effective potential shown in Fig. 2 and it is given by

$$U_q(\alpha_m)\equiv U(q,\alpha_m)=\frac{4\pi}{q^2}+\frac{4\pi}{q^2}\sum_s e_s^2\frac{1}{V}\sum_{\mathbf{p}}\frac{1}{\beta}$$
$$\times\sum_l G_{\mathbf{p}+\mathbf{q}/2}(\zeta_l+\alpha_m,s)G_{\mathbf{p}-\mathbf{q}/2}(\zeta_l,s)U_{\mathbf{q}}(\alpha_m),\qquad(23)$$

where $\alpha_m=i2\pi m(1/\beta)$, $m=0,\pm1,\pm2,\cdots$. We can now cast Eq. (23) into the form

$$U_q(\alpha_m)=\frac{4\pi}{q^2}\left[1-\frac{4\pi}{q^2}\sum_s e_s^2Q_q(\alpha_m,s)\right]^{-1},\qquad(24)$$

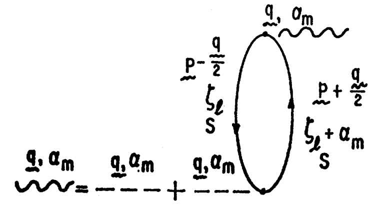

FIG. 2

FIG. 2. The integral equation for the effective interaction.

where

$$Q_q(\alpha_m,s)=\frac{1}{V}\sum_p\frac{1}{\beta}\sum_l G_{\mathbf{p}+\mathbf{q}/2}(\zeta_l+\alpha_m,s)G_{\mathbf{p}-\mathbf{q}/2}(\zeta_l,s)$$
$$=\frac{1}{(2\pi)^3}\int d\mathbf{p}\,\frac{n_{\mathbf{p}+\mathbf{q}/2}(s)-n_{\mathbf{p}-\mathbf{q}/2}(s)}{\epsilon_{\mathbf{p}+\mathbf{q}/2}(s)-\epsilon_{\mathbf{p}-\mathbf{q}/2}(s)-\alpha_m}\qquad(25)$$

and

$$n_p(s)=[e^{\beta\epsilon_p(s)-\beta\mu_s}+1]^{-1},\qquad(26)$$

is the Fermi distribution of the s species.

We now calculate the contribution of the diagrams, given in Fig. 1 for the case $\mathbf{k}=0$. We assert that these are the leading asymptotic terms for long wavelength. Using the prescription of Luttinger and Ward[6] for the many species system under consideration we obtain

$$M_{\mu\nu}(\omega_n)\equiv M_{\mu\nu}(\mathbf{k}=0,\omega_n)=\frac{\hbar^2}{V}\sum_{s,s'}\frac{e_se_{s'}}{m_sm_{s'}}\sum_{\mathbf{p},\mathbf{p}'}p_\mu p_\mu'$$
$$\times\sum_{i=1}^{5}K_{\mathbf{pp}'}^{(i)}(\omega_n,s,s'),\qquad(27)$$

where $K_{\mathbf{pp}'}^{(i)}(\omega_n,s,s')$ corresponds to the ith diagram of Fig. 1

$$K_{\mathbf{pp}'}^{(1)}(\omega_n,s,s')=\delta_{ss'}e_s^2\frac{1}{V}\frac{1}{\beta}\sum_m U_{\mathbf{p}-\mathbf{p}'}(\alpha_m)\frac{1}{\beta}\sum_l G_{\mathbf{p}}(\zeta_l,s)G_{\mathbf{p}}(\zeta_l+\omega_n,s)G_{\mathbf{p}'}(\zeta_l+\omega_n+\alpha_m,s)G_{\mathbf{p}'}(\zeta_l+\alpha_m,s),$$

$$K_{\mathbf{pp}'}^{(2)}(\omega_n,s,s')=\delta_{ss'}\delta_{\mathbf{pp}'}e_s^2\frac{1}{V}\sum_{\mathbf{q}}\frac{1}{\beta}\sum_m U_{\mathbf{q}}(\alpha_m)\frac{1}{\beta}\sum_l[G_{\mathbf{p}}(\zeta_l,s)]^2G_{\mathbf{p}-\mathbf{q}}(\zeta_l-\alpha_m,s)G_{\mathbf{p}}(\zeta_l-\omega_n,s),$$

$$K_{\mathbf{pp}'}^{(3)}(\omega_n,s,s')=\delta_{ss'}\delta_{\mathbf{pp}'}e_s^2\frac{1}{V}\sum_{\mathbf{q}}\frac{1}{\beta^2}\sum_{m,l}U_{\mathbf{q}}(\alpha_m)[G_{\mathbf{p}}(\zeta_l,s)]^2G_{\mathbf{p}+\mathbf{q}}(\zeta_l+\alpha_m,s)G_{\mathbf{p}}(\zeta_l+\omega_n,s),$$

$$K_{\mathbf{pp}'}^{(4)}(\omega_n,s,s')=e_s^2e_{s'}^2\frac{1}{V^2}\sum_{\mathbf{q}}\frac{1}{\beta}\sum_m U_q(\alpha_m)U_q(\alpha_m-\omega_n)\frac{1}{\beta}\sum_l G_{\mathbf{p}}(\zeta_l,s)G_{\mathbf{p}}(\zeta_l-\omega_n,s)\qquad(28)$$
$$\times G_{\mathbf{p}-\mathbf{q}}(\zeta_l-\alpha_m,s)\frac{1}{\beta}\sum_{l'}G_{\mathbf{p}'}(\zeta_{l'},s')G_{\mathbf{p}'}(\zeta_{l'}-\omega_n,s')G_{\mathbf{p}'-\mathbf{q}}(\zeta_{l'}-\alpha_m,s'),$$

$$K_{\mathbf{pp}'}^{(5)}(\omega_n,s,s')=e_s^2e_{s'}^2\frac{1}{V^2}\sum_l\frac{1}{\beta}\sum_m U_q(\alpha_m)U_q(\alpha_m-\omega_n)\frac{1}{\beta}\sum_{\mathbf{q}}G_{\mathbf{p}}(\zeta_l,s)$$
$$\times G_{\mathbf{p}}(\zeta_l-\omega_n,s)G_{\mathbf{p}-\mathbf{q}}(\zeta_l-\alpha_m,s)\frac{1}{\beta}\sum_{l'}G_{\mathbf{p}'}(\zeta_{l'},s')G_{\mathbf{p}'}(\zeta_{l'}+\omega_n,s')G_{\mathbf{p}'+\mathbf{q}}(\zeta_{l'}+\alpha_m,s').$$

We now carry out the summation over l and l' by converting the sums into integrals (see LW[6]). After considerable manipulations (see an example in Appendix A) we obtain

$$M_{\mu\nu}(\omega_n)=-\frac{1}{2}\int\frac{d\mathbf{q}}{(2\pi)^3}q_\mu q_\nu\sum_{s,s'}e_s^2e_{s'}^2\times\frac{e_s}{m_s}\left(\frac{e_s}{m_s}-\frac{e_{s'}}{m_{s'}}\right)F_{\mathbf{q}}(\omega_n,s,s'),\quad(29)$$

where

$$F_{\mathbf{q}}(\omega_n,s,s')=\frac{\hbar^2}{\omega_n^2}\frac{1}{\beta}\sum_m U_q(\alpha_m)U_q(\alpha_m+\omega_n)\times[Q_q(\alpha_m+\omega_n,s)-Q_q(\alpha_m,s)]\times[Q_q(\alpha_m+\omega_n,s')-Q_q(\alpha_m,s')].\quad(30)$$

Equations (29) and (30) are then our generalization of Eq. (15) of PE to a system of arbitrary number of species. We wish to point out that without this generalization one would be led to the incorrect result of PE, for finite-mass ratio [see PE Eqs. (19)–(22)]. It is easily seen that in Eq. (29) the sum over s' runs only on species different from s. This indicates that for a system of one species (with smeared-out background) the real part of the absorbtion coefficient vanishes and the imaginary part is given by Eq. (7) only.

Equations (29) and (30) are not suitable for the analytical continuation to the upper z half-plane. In order to perform the continuation one has first to evaluate the summation over m. We adopt essentially a method developed by PE to carry out this summation and for the benefit of the reader it is given in Appendix B. We now use Eq. (B8) and analytically continued Eq. (30) to obtain

$$F_{\mathbf{q}}^+(\omega,s,s')=\frac{1}{\omega^2}\frac{P}{i\pi}\int dx\,H(x)\{U_q^+(x)U_q^+(x+\hbar\omega)[Q_q^+(x+\hbar\omega,s)-Q_q^+(x,s)][Q_q^+(x+\hbar\omega,s')-Q_q^+(x,s)]-U_q^-(x)U_q^+(x+\hbar\omega)[Q_q^+(x+\hbar\omega,s)-Q_q^-(x,s)][Q_q^+(x+\hbar\omega,s')-Q_q^-(x,s')]\}.\quad(30)$$

In Eq. (30) we denote by P the principal value,

$$H(x)=[e^{\beta x}-1]^{-1},\quad(31)$$

and

$$Q_q^\pm(x,s)=\frac{1}{(2\pi)^3}\int d\mathbf{p}\frac{n_{\mathbf{p}+\mathbf{q}/2}(s)-n_{\mathbf{p}-\mathbf{q}/2}(s)}{\epsilon_{\mathbf{p}+\mathbf{q}/2}(s)-\epsilon_{\mathbf{p}-\mathbf{q}/2}(s)-x\mp i\epsilon}.\quad(32)$$

$U_q^\pm(x)$ is determined by Eq. (24), replacing $Q_q(\alpha_m,s)$ by $Q_q^\pm(x,s)$.

Now we make use of Eqs. (7), (12), (29), (30), and (31) to obtain our final result for the complex conductivity.

$$\sigma(\omega)=\sigma_0(\omega)+i\frac{4\pi}{3\omega^3}\sum_{s,s'}e_s^2e_{s'}^2\frac{e_s}{m_s}\left(\frac{e_s}{m_s}-\frac{e_{s'}}{m_{s'}}\right)\frac{1}{(2\pi)^3}\int dq\,q^4\frac{P}{2\pi i}\int dx[e^{\beta x}-1]^{-1}\times\{U_q^+(x)U_q^+(x+\hbar\omega)[Q_q^+(x+\hbar\omega,s)-Q_q^+(x,s)][Q_q^+(x+\hbar\omega,s')-Q_q^+(x,s')]-U_q^-(x)U^+(x+\hbar\omega)[Q_q^+(x+\hbar\omega,s)-Q_q^-(x,s)][Q_q^+(x+\hbar\omega,s')-Q_q^-(x,s')]\}.\quad(33)$$

In the last equation use has been made of the fact that Q_q depends only on the absolute value of q, and that $\sigma_{\mu\nu}(\omega)=\delta_{\mu\nu}\sigma(\omega)$. This result is rather complicated but in principle can be evaluated analytically or numerically for specific problems. It is clear that the result is applicable for both classical and quantum plasmas for any mass ratio of the species of the system, and for temperatures, where the average potential energy of interaction per particle is smaller than the average kinetic energy.

IV. SPECIAL CASES

In this section we discuss some interesting special cases where Eq. (33) can be cast in simpler forms.

A. Classical Plasmas

We divide our classical result into two kinds: (i) strictly classical and (ii) semiclassical. By strictly classical we understand results which are independent of h, and by semiclassical we understand inclusion of some h-dependent terms which can prevent in some specific situations the divergences caused by large-angle collisions.

In order to obtain the strictly classical limit of Eq. (33) we first make the transformation $x=huq$, and then $Q_q^\pm(u,s)$ becomes

$$Q_q^\pm(u,s)=\frac{n_s}{m_s}\int d\mathbf{v}\frac{\mathbf{q}\cdot(\partial/\partial\mathbf{v})f_0(\mathbf{v},s)}{\mathbf{q}\cdot\mathbf{v}-qu\mp i\epsilon}=-n_s\beta[1+uf_s^\pm(u)].\quad(34)$$

In Eq. (34) use has been made of the following transformations

$$\mathbf{p}\rightarrow m_s\mathbf{v}/\hbar,$$

$$n_{\mathbf{p}+\mathbf{q}/2}(s)-n_{\mathbf{p}-\mathbf{q}/2}(s)\rightarrow\left(\frac{2\pi\hbar}{m_s}\right)^3n_s\frac{\hbar}{m_s}\mathbf{q}\cdot\frac{\partial f_0(\mathbf{v},s)}{\partial\mathbf{v}},\quad(35)$$

where now $f_0(\mathbf{v},s)$ is the Maxwell distribution for the s species, and

$$f_s^{\pm}(u)\equiv\int du'\frac{f_s(u')}{u'-u\mp i\epsilon}, \tag{36}$$

with $f_s(u)$ the one-dimensional distribution. We also introduce the classical dielectric constant

$$\rho(q,u)=1+\int\frac{du'}{u'-u-i\epsilon}\sum_s(-1)\frac{\omega_s^2}{q^2}\frac{\partial}{\partial u'}f_s(u'), \tag{37}$$

and thus

$$U_q^+(u)\rightarrow\frac{4\pi}{q^2}\frac{1}{\rho(q,u)}. \tag{38}$$

In the classical limit

$$[H(x)]^{-1}=\epsilon^{\beta x}-1=\epsilon^{\beta q\hbar u}-1\rightarrow\beta q\hbar u. \tag{39}$$

Thus,

$$\sigma(\omega)=\sigma_0(\omega)+i\frac{4\pi}{\omega^3}\sum_{s,s'}e_s\omega_s^2K_{s'}^2\left(\frac{e_s}{m_s}-\frac{e_{s'}}{m_{s'}}\right)\frac{1}{(2\pi)^3}\int_0^{q_{\max}}dq\frac{P}{2\pi i}\int_{-\infty}^{\infty}\frac{du}{u}\frac{1}{\rho(q,\,u+w)}$$
$$\times\left\{\frac{1}{\rho(q,u)}[(u+w)f_s^+(u+w)-uf_s^+(u)][(u+w)f_{s'}^+(u+w)-uf_{s'}^+(u)]\right.$$
$$\left.-\frac{1}{\rho^*(q,u)}[(u+w)f_s^+(u+w)-uf_s^-(u)][(u+w)f_{s'}^+(u+w)-uf_{s'}^-(u)]\right\}. \tag{40}$$

Equation (40) is a trivial generalization of the result given by Oberman *et al.*[2] for any number of species. In Eq. (40)

$$K_s^2=4\pi e_s^2n_s\beta \tag{41}$$

is the square of the reciprocal of the Debye radius for the s species, $q_{\max}$ is the cutoff,[9] and $w=\omega/q$. A further simplification of Eq. (40) can be achieved by the following procedure. The first part of the integral of Eq. (40) is analytic in the entire upper half-plane except a pole at $u=0$ and can be, therefore, integrated easily. In the second part of the integrand we make the transformation

$$\frac{1}{u}\frac{1}{\rho^*(q,u)\rho(q,\,u+w)}(u+w)f_s^+(u+w)[(u+w)f_{s'}^+(u+w)-uf_{s'}^-(u)]\rightarrow$$
$$\frac{1}{u+w}\frac{1}{\rho^*(q,u)\rho(q,\,u+w)}uf_s^-(u)[(u+w)f_{s'}^+(u+w)-uf_{s'}^-(u)],$$

to obtain

$$\sigma(\omega)=\sigma_0(\omega)+i\frac{4\pi}{3\omega^3}\sum_{s,s'}e_s\omega_s^2K_s^2\left(\frac{e_s}{m_s}-\frac{e_{s'}}{m_{s'}}\right)\frac{1}{(2\pi)^3}\int dq\frac{w}{\rho(q,0)}$$
$$\times\left\{\frac{w}{2}\frac{f_s^+(w)f_{s'}^+(w)}{\rho(q,w)}-\rho(q,0)\frac{P}{2\pi i}\int du\frac{uf_s^-(u)}{\rho^*(q,u)\rho(q,\,u+w)}\left[\frac{f_{s'}^+(u+w)}{u}-\frac{f_{s'}^-(u)}{u+w}\right]\right\}, \tag{42}$$

which can be easily identified with Eq. (54) of Oberman *et al.*[2] for the hydrogenic plasma, and thus we are in complete agreement with their results.

We now turn to a brief discussion of the semiclassical case. Here $Q^{\pm}$ of Eq. (34) is replaced by

$$Q_q^{\pm}(u,s)=e^{-\hbar^2q^2\beta/8)m_s}\frac{n_s}{m_s}\int d\mathbf{v}\frac{\mathbf{q}\cdot(\partial/\partial\mathbf{v})f_0(\mathbf{v},s)}{\mathbf{q}\cdot\mathbf{v}-qu\mp i\epsilon}, \tag{43}$$

where in the passage to the classical limit of $n_{p+q/2}(s)$ of Eq. (35) we keep in the exponential the $\hbar^2q^2$ term. The persistence of this term would lead to an inclusion of a weighting function of the type $e^{-\hbar^2q^2\beta/8m}$ in the integrands of Eqs. (40) and (42). For the specific case of hydrogenic plasma with infinite ion mass we obtain[10]

$$\sigma(\omega)=\sigma_0(\omega)\left\{1-\frac{2}{3\pi}\frac{e^2}{m\omega^2}\int_0^{\infty}dq\,e^{-\hbar^2q^2\beta/8m}q^4\frac{q^2+K^2}{q^2+2K^2}\right.$$
$$\left.\times\left[\frac{1}{q^2+K^2}-\frac{1}{q^2D(q,\omega)}\right]\right\}, \tag{44}$$

where m and e stand for the electron mass and charge, K^2 is given by Eq. (44) and $D(q,\omega)$ is the dielectric constant given in Eq. (37) for system of electrons only. We wish to point out that for the real part of $\sigma(\omega)$ the integral over q diverges logarithmically in the strictly classical case, and therefore, a cutoff $q_{\max}=(e^2\beta)^{-1}$ is in-

[9] L. Spitzer, Jr., *Physics of Fully Ionized Gases* (Interscience Publishers, Inc., New York, 1956), p. 76.

[10] This equation agrees in the strictly classical case ($\hbar=0$) with Eqs. (58) and (59) of reference 2.

troduced (see discussion of this point in reference 9). In the semiclassical case although the integral converges, the inclusion of large-angle collisions may render another effective cutoff to be invoked for essentially smaller values of q. Equation (44), as it stands, is rigorously valid when the Born approximation for electron-ion scattering is valid, namely, when $kT=1/\beta\ll 1$ Rydberg.

B. Quantum Plasmas

The general expression for the high-frequency conductivity for quantum systems is given by Eq. (33). In this section we restrict ourself to a specific problem of ion-electron quantum plasma where the ions are randomly distributed fixed scatterers. This treatment may find its application in some solid-state problems, which we intend to discuss in a future communication.

We rewrite Eq. (33) for the case of an ion-electron system, where all the terms which depend explicitly on the inverse ion mass are omitted,

$$\begin{aligned}\sigma(\omega)=\sigma_0(\omega)+i\frac{8}{3\omega^3}\frac{e^6}{m^2}\int dq\frac{P}{2\pi i}\\ \times\int dx(e^{\beta x}-1)^{-1}\frac{1}{\epsilon(q,\,x+\hbar\omega)}\\ \times\bigg\{\frac{1}{\epsilon(q,x)}[Q_q^+(x+\hbar\omega)-Q_q^+(x)]\\ \times[\tilde{Q}_q^+(x,\hbar\omega)-\tilde{Q}_q^+(x)]\\ -\frac{1}{\epsilon^*(q,x)}[Q_q^+(x+\hbar\omega)-Q_q^-(x)]\\ \times[\tilde{Q}_q^+(x+\hbar\omega)-\tilde{Q}_q^-(x)]\bigg\}.\end{aligned}\quad(45)$$

Here

$$\epsilon(q,x)=1-\frac{4\pi}{q^2}e^2[Q_q^+(x)+\tilde{Q}_q^+(x)]\quad(46)$$

stands for the usual retarded dielectric constant of the ion-electron system and $Q_q(x)$ and $\tilde{Q}_q(x)$ represent, respectively, the electron and ion Q's of Eq. (32).

In order to carry out the limit of fixed-ion scatterers we treat the ions classically, namely, we replace the Q's by their classical representation, Eq. (34), and then prescribe for the ions

$$f_i^{\pm}(u)=-P\frac{1}{u}\pm i\pi\delta(u)\quad(47)$$

[namely, substitute $f_i(u)=\delta(u)$ in Eq. (36)]. One should notice now that in this limit

$$\epsilon(q,x)\rightarrow P(q,x)=1-\frac{4\pi e^2}{q^2}Q_q^+(x),\quad(48)$$

while

$$(e^{\beta x}-1)[\tilde{Q}_q^+(x+\hbar\omega)-\tilde{Q}_q^{\pm}(x)]\rightarrow\pm i\pi n\delta(x),\quad(49)$$

where n is the average electron or ion density. Thus, we obtain

$$\begin{aligned}\sigma(\omega)&=\sigma_0(\omega)+i\frac{8}{3\omega^3}\frac{e^6}{m^2}n\int dq\frac{Q_q^+(\hbar\omega)-Q_q(0)}{P(q,0)P(q,\hbar\omega)}\\&=\sigma_0(\omega)\bigg[1-\frac{2}{3\pi}\frac{e^2}{m\omega^2}I(\omega)\bigg],\end{aligned}\quad(50)$$

where

$$Q_q(0)=\tfrac{1}{2}[Q_q^+(0)+Q_q^-(0)],$$

and

$$I(\omega)=\int dq\,q^2\bigg(\frac{1}{P(q,0)}-\frac{1}{P(q,\hbar\omega)}\bigg).\quad(51)$$

This result is similar in its form to the classical result obtained by Dawson and Oberman.[1]

V. DISCUSSION

In this paper we have derived a general expression of the absorption coefficient of electromagnetic waves in quantum and classical plasma. Our result, Eq. (33), is valid for multispecies systems of charged particles. With the restriction to applied fields of high-frequency and long wavelength, we have properly accounted for collective effects. We contend that these results represent the most complete and systematic expression for the conductivity [hence, immediately for the absorption coefficient of plasmas, see reference 1]. The classical limit of our result is in complete agreement with the previous results of references 1 and 2 and we refer the reader to these papers for further discussion of the classical case. An interesting new result is our expression for the conductivity of a quantum system of electrons in the presence of heavy scatterers as given by Eq. (51). We reserve the discussion of this result and its possible implication for solid-state systems to subsequent publication.

In comparing our classical and semiclassical results with those of PE[3] we conclude that their results for *finite* ratio of the electron-ion mass [see Eqs. (19)–(22) of reference 3] are incorrect but that their final result for infinitely heavy ions [PE Eq. (24)] can be cast into our result given by Eq. (44) [see reference 2, Appendix B for the evaluation of the integrals given in PE Eq. (24)]. However, there is a point which we would like to clarify as much as we can, that is, in principle, we disagree with the result of PE. PE have performed their mass-limiting procedure in several stages. From the beginning, up to Eq. (22) of their paper, they have neglected all terms *explicitly* proportional to the mass ratio; this amounts to the *direct* contribution of the ions to the current. From this point on one must be extremely cautious and realize, especially in performing velocity integrals, that the mass ratio is not the only quantity which can be small in the problem. If the mass ratio is considered small but *finite* then *all* integrals must be performed first and then the limit taken. If this course is followed one obtains results different from PE, see reference 2, Eq. (71). It seems to us that this is the course that PE thought they pursued but, in fact, did not. The result they do obtain is that for a plasma *model* in which massive ions are thrown into an electron plasma, not at random, but respecting the thermal-equilibrium Debye correlation for finite temperature, but having *no dynamics*. This result can be obtained by doing the mass ratio limit first, and then performing velocity integrals, as was shown in the classical treatment in reference 2, and in the present work.

A comparison of our classical results with the results of Dubois, Gilinsky and Kivelson as given in their letter[4] is difficult to make. However, from a study of their more complete report[4] [see Eq. (6.20) of the report] we are led to the conclusion, in agreement with references 1 and 2, that their result is incorrect. This is probably due to their omission of an important class of diagrams.

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APPENDIX A.

We wish to represent here an example of the calculation of one diagram of Fig. 1, namely diagrams 1–4. To calculate the contribution of this diagram to $M_{\mu\nu}(\omega_n)$ we first substitute[4] K of Eq. (28) into Eq. (27) and replace G_p by its representation, Eq. (22);

$$M_{\mu\nu}^{(4)}(\omega_n)=\frac{\hbar^2}{V^3}\sum_{s,s'}\frac{e_s^3e_{s'}^3}{m_sm_{s'}}\sum_{\mathbf{p},\mathbf{p}'}p_\mu p_\nu'\sum_{\mathbf{q}}\frac{1}{\beta}\sum_m U_q(\alpha_m)$$
$$\times U_q(\alpha_m-\omega_n)L_{\mathbf{p}}(\alpha_m,\omega_n,s)L_{\mathbf{p}'}(\alpha_m,\omega_n,s'),\quad\text{(A1)}$$

where

$$L_{\mathbf{p}}(\alpha_m,\omega_n,s)=\frac{1}{\beta}\sum_l\frac{1}{\zeta_l(s)-\epsilon_p(s)}\frac{1}{\zeta_l(s)-\omega_n-\epsilon_p(s)}$$
$$\times\frac{1}{\zeta_l(s)-\alpha_m-\epsilon_{p-q}(s)}.\quad\text{(A2)}$$

Now to perform the summation we replace ζ_l by ζ (see LW) and write L_p as an integral along a path C given in Fig. 3

$$L_{\mathbf{p}}(\alpha_m,\omega_n,s)=\frac{1}{2\pi i}\int_C d\zeta\,[e^{\beta(\zeta-\mu_s)}+1]^{-1}\frac{1}{\zeta-\epsilon_p(s)}$$
$$\times\frac{1}{\zeta-\omega_n-\epsilon_p(s)}\frac{1}{\zeta-\alpha_m-\epsilon_{\mathbf{p}-\mathbf{q}}(s)}.\quad\text{(A3)}$$

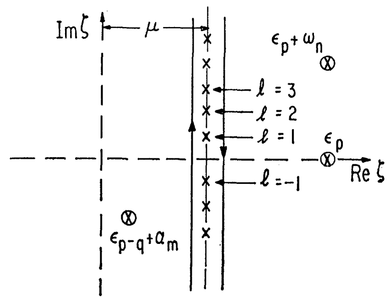

FIG. 3. The contour of integration used in Eq. (A3).

This integral can be evaluated by Cauchy's theorem to give

$$\begin{aligned}L_{\mathbf{p}}(\alpha_m,\omega_n,s)&\\=[e^{\beta(\epsilon_p-\mu_s)}+1]^{-1}&[-\omega_n]^{-1}[-\alpha_m-\epsilon_{\mathbf{p}-\mathbf{q}}(s)+\epsilon_p(s)]^{-1}\\&+[e^{\beta[\epsilon_p(s)+\omega_n-\mu_s]}+1]^{-1}[\omega_n]^{-1}\\&\times[\omega_n-\alpha_m-\epsilon_{\mathbf{p}-\mathbf{q}}(s)+\epsilon_p(s)]^{-1}\\&+[e^{\beta[\epsilon_{\mathbf{p}-\mathbf{q}}(s)+\alpha_m-\mu_s]}+1]^{-1}\\&\times[-\omega_n+\alpha_m+\epsilon_{\mathbf{p}-\mathbf{q}}(s)+\epsilon_p(s)]^{-1}\\&\qquad\times[\alpha_m-\epsilon_{\mathbf{p}-\mathbf{q}}(s)+\epsilon_p(s)]^{-1}.\quad\text{(A4)}\end{aligned}$$

If we now realize that

$$e^{\beta\omega_n}=e^{\beta\alpha_m}=e^{i2\pi}=1,$$

and use the definition of Eq. (26), we get

$$L_{\mathbf{p}}(\alpha_m,\omega_n,s)=\frac{1}{\omega_n}[n_{\mathbf{p}-\mathbf{q}}(s)-n_p(s)]$$
$$\times\{[\alpha_m-\omega_n-\epsilon_p(s)+\epsilon_{\mathbf{p}-\mathbf{q}}(s)]^{-1}$$
$$-[\alpha_m-\epsilon_p(s)-\epsilon_{\mathbf{p}-\mathbf{q}}(s)]^{-1}\},\quad\text{(A5)}$$

and thus

$$M_{\mu\nu}^{(4)}(\omega_n)=\frac{\hbar^2}{\omega_n^2}\sum_{s,s'}\frac{e_s^3e_{s'}^3}{m_sm_{s'}}\frac{1}{(2\pi)^3}\int d\mathbf{q}\,\frac{1}{\beta}\sum_m U_q(\alpha_m)$$
$$\times U_q(\alpha_m-\omega_n)R_\mu(\alpha_m,\omega_n,s)R_\nu(\alpha_m,\omega_n,s).\quad\text{(A6)}$$

and

$$R_\mu(\alpha_m,\omega_n,s)=\frac{1}{(2\pi)^3}\int d\mathbf{p}\,(p_\mu+\tfrac{1}{2}q_\mu)$$
$$\times[n_{\mathbf{p}+\mathbf{q}/2}(s)-n_{\mathbf{p}-\mathbf{q}/2}(s)]$$
$$\times[(\alpha_m-\omega_n-\Delta)^{-1}-(\alpha_m-\Delta)^{-1}],\quad\text{(A7)}$$

where

$$\Delta=\epsilon_{\mathbf{p}+\mathbf{q}/2}(s)-\epsilon_{\mathbf{p}-\mathbf{q}/2}(s),$$

and $V^{-1}\sum_{\mathbf{q}}$ was replaced by $1/(2\pi)^3\int d\mathbf{q}$. If we use Eq. (25) we obtain

$$R_\mu(\alpha_m,\omega_n,s)=\tfrac{1}{2}q_\mu[Q_q(\alpha_m-\omega_n,\,s)-Q_q(\alpha_m,s)],\quad\text{(A8)}$$

and finally

$$M_{\mu\nu}^{(4)}(\omega_n)=\frac{1}{4}\frac{\hbar^2}{\omega_n^2}\sum_{s,s'}\frac{e_s^3e_{s'}^3}{m_sm_{s'}}\frac{1}{(2\pi)^3}\int d\mathbf{q}\,q_\mu q_\nu\frac{1}{\beta}\sum_m U_q(\alpha_m)$$
$$\times U_q(\alpha_m-\omega_n)[Q_q(\alpha_m-\omega_n,\,s)-Q_q(\alpha_m,s)]$$
$$\times[Q_q(\alpha_m-\omega_n,\,s')-Q_q(\alpha_m,s')].\quad\text{(A9)}$$

APPENDIX B.

In this Appendix we wish to describe briefly PE's method of evaluating the sum

$$f(\omega_n)=\frac{1}{\beta}\sum_m \varphi(\alpha_m)\psi(\alpha_m+\omega_n)$$
$$\alpha_m=i2\pi m/\beta \quad m=0,\pm1,\pm2,\cdots$$
$$\omega_n=i2\pi m/\beta \quad n=1,2,3,\cdots, \qquad \text{(B1)}$$

where $\varphi(\alpha_m)$ and $\psi(\alpha_m)$ are either our $Q_q(\alpha_m)$ or $U_q(\alpha_m)$. Define $\varphi^R(\alpha)$ and $\psi^R(\alpha)$ as the analytical continuation of $\varphi(\alpha_m)$ and $\psi(\alpha_m)$, respectively, to the upper half-plane, and $\varphi^A(\alpha)$, $\psi^A(\alpha)$ to the lower half-plane. It is easily seen that if at least one of the functions φ or ψ coincide with Q the analytical continuation of $\varphi\psi$ goes to zero at infinity at least as α^{-2}; and both

$$\varphi(\alpha)\equiv\begin{cases}\varphi^R(\alpha) & \text{for} \quad \mathrm{Im}\alpha>0\\ \varphi^A(\alpha) & \text{for} \quad \mathrm{Im}\alpha<0\end{cases}$$

and

$$\psi(\alpha)\equiv\begin{cases}\psi^R(\alpha) & \text{for} \quad \mathrm{Im}\alpha>0\\ \psi^A(\alpha) & \text{for} \quad \mathrm{Im}\alpha<0,\end{cases} \qquad \text{(B2)}$$

have a cut along the real axis of α.

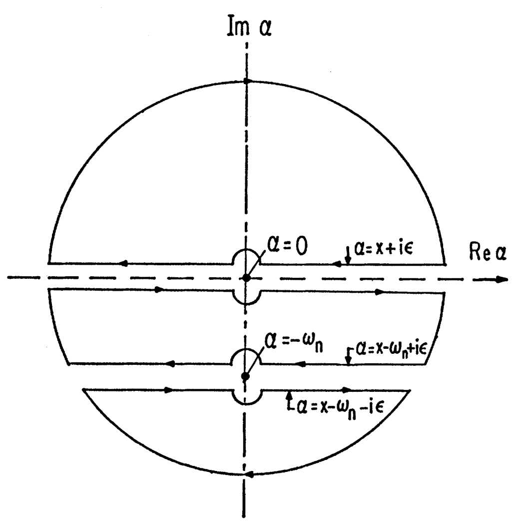

FIG. 4. The contour of integration used in Eq. (B4).

If we introduce the function

$$H(\alpha)=(e^{\beta\alpha}-1)^{-1} \qquad \text{(B3)}$$

which has poles at the points $\alpha_m=i2\pi m/\beta$, and the contour C in the α plane (see Fig. 4), we are able to represent $f(\omega_n)$ as

$$f(\omega_n)=-\frac{1}{2\pi i}\int_C d\alpha\, H(\alpha)\varphi(\alpha)\psi(\alpha+\omega_n)$$
$$\times\frac{1}{\beta}[\varphi(0)\psi(\omega_n)+\varphi(-\omega_n)\psi(0)]. \qquad \text{(B4)}$$

To proceed we carry out the integration of (B4), noticing that the large circle does not contribute to the integral, and that the integrals along the small semicircles around $\alpha=0$ and $\alpha=-\omega_n$ just cancel the last term on the right-hand side of Eq. (B4). Thus, we get

$$f(\omega_n)=\frac{P}{2\pi i}\int_{-\infty}^{\infty}dx$$
$$\times H(x)\{[\varphi^R(x+i\epsilon)-\varphi^A(x-i\epsilon)]\psi^R(x+\omega_n)$$
$$+\varphi^A(x-\omega_n)[\psi^R(x+i\epsilon)-\psi^A(x-i\epsilon)]\}, \qquad \text{(B5)}$$

where P stands for the principal value and $\epsilon\to 0^+$. If we now take into account the fact that x is real and ω_n is imaginary we easily establish the relation

$$\varphi^A(x-\omega_n)=\varphi^R(x+\omega_n), \qquad \text{(B6)}$$

and with Eq. (B2) and the definition of Eq. (13), we obtain

$$f(\omega_n)=\frac{P}{2\pi i}\int dx\, H(x)\{[\varphi^+(x)-\varphi^-(x)]\psi^R(x+\omega_n)$$
$$+\varphi^R(x+\omega_n)[\psi^+(x)-\psi^-(x)]\}. \qquad \text{(B7)}$$

Equation (B7) is our required result and it can be generalized to any combination of Q's and U's provided that at least one Q is present.

It is now easy to continue $f(\omega_n)$ to the upper half-plane of ω and to get finally

$$f(\omega)=\frac{P}{2\pi i}\int dx\, H(x)\{[\varphi^+(x)-\varphi^-(x)]\psi^+(x+\hbar\omega)$$
$$+\varphi^+(x+\hbar\omega)[\psi^+(x)-\psi^-(x)]\}. \qquad \text{(B8)}$$